The Binomial Theorem

(and Pascal's Triangle) Lesson 9.5

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Observations

$(x+y)^n$ n+1 terms

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's $\triangle$

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

Find the Binomial Coefficients

$${}_{n}C_{r} = \frac{n!}{(n-r)!r!}$$

1. ${}_{10}C_{5}$ $\quad \frac{10!}{(10-5)!\,5!} \quad \frac{10!}{5!5!} \quad \frac{\cancel{10}^{\,}\cdot \cancel{9}^{3}\cdot \cancel{8}^{2}\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{5!}}$ $\quad 3\cdot 2\cdot 7\cdot 6$ $\quad 252$

2. $\binom{8}{2}$ $n=8$, $r=2$ $= 28$

3. ${}_{12}C_{4}$ $\quad \frac{12!}{8!\,4!} = 495$

4. ${}_{12}C_{8}$ $\quad \frac{12!}{4!\,8!} = 495$

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Find the expansion of $(x + 2)^4$.

$(x + 2)^4 = \sum_{r=0}^{4} {}_4C_r x^{4-r} 2^r$

${}_4C_0\, x^4 (2)^0 + {}_4C_1\, x^{4-1} 2^1 + {}_4C_2\, x^2 (2)^2 + {}_4C_3\, x^1 2^3 + {}_4C_4\, x^0 2^4$

$\frac{4!}{4!\,0!} x^4 + \frac{4!}{3!\,1!} x^3 (2) + \frac{4!}{2!\,2!} x^2 (4) + \frac{4!}{1!\,3!} x^1 (8)\ \frac{4!}{0!\,4!} (16)$

$1x^4 + 8x^3 + 24x^2 + 32x + 16$

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Find the expansion of $(3x - y)^4$.

$(3x - y)^4 =$

$$\sum_{r=0}^{4} {}_nC_r \, (3x)^{n-r}(-y)^r$$

$$\frac{4!}{1!\,3!}\, 3x(-y)^3$$

$${}_4C_0\,(3x)^4(-y)^0 + \underline{{}_4C_1}\,(3x)^3(-y)^1 + {}_4C_2(3x)^2(-y)^2 + {}_4C_3\,(3x)^1\,(-y)^3 +$$

$$81x^4 + \frac{4!}{3!\,1!}(27x^3)(-y) + \frac{4!}{2!\,2!}^{6}\, 9x^2y^2 \qquad {}_4C_4\,(3x)^0\,(-y)^4$$

$$\frac{4!}{1!\,4!}$$

$$81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$$

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Find the coefficient of x^8 in the expansion of $(x^2 + 2)^{12}$.

$n=$

$r=4$

$(x^2)^{12-r} = x^8$

$x^{24-2r} = x^8$

$24 - 2r = 8$

$-2r = -16$

$r = 8$

${}_{12}C_8 (x^2)^{12-8} (2)^8$

$\frac{12!}{4!\,8!} x^8 (256)$

$\frac{\cancel{12} \cdot 11 \cdot \cancel{10}^{5} \cdot 9 \cdot \cancel{8!}}{\cancel{4} \cdot 3 \cdot \cancel{2} \cdot 1 \; \cancel{8!}}$

$11 \cdot 5 \cdot 9 (256) = 126{,}720$

Find the fifth term of the expansion of $(2x - 3y)^{10}$.

Section 9.5 p. 649; 19-34 x 3's, 45-60 x 3's